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Energy saving in chaotic laminar mixing

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Abstract—Chaotic mixing in laminar flows at high Schmidt numbers is considered: the time taken for molecular diffusion to erase the concentration gradient created by advection is estimated. The energy dissipated during mixing in a Stokes flow is calculated, and compared with that for a turbulent flow. It is shown that, at high Schmidt numbers, laminar mixing is more energy-efficient than turbulent mixing. A parameter is then introduced that takes into account both the time spent in mixing and the energy dissipated during this time: in fact, it is the product of time taken and energy dissipated. Again, at high Schmidt numbers, a Stokes flow is shown to be more efficient even when judged by this parameter. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

As is well known, mixing implies not only a stirring process, which transports mechanically fluid particles and distributes them more uniformly, but also a diffusion process, which smooths the concentration gradients created by advection. Mixing is thus a difficult problem from its very definition. However, as difficult a problem is the 'how', and when asked 'how to mix', the fluid mechanic is tempted to answer 'by making the flow turbulent'. After all, a turbulent flowfield is three-dimensional and random, which allows efficient stirring, and creates very thin structures that further the effects of molecular diffusion. However, is this choice always the best? If it is supposed, for example, that the fluids that must be mixed are very viscous, then making the flow turbulent implies a large energy dissipation rate. On the other hand, the energy dissipation required in a Stokes flow is very small, but full mixing takes longer.

The fact that a laminar flow can generate efficient stirring thanks to chaotic trajectories has been studied a lot [1–9]. Stirring in a Stokes flow between two eccentric cylinders has been considered both experimentally and numerically [10–15]. Curiously, the effects of molecular diffusion have only been studied in a few papers [14, 15, 16–18], in almost all of which—except for that by Ott and Antonsen [17]—the kinematics of a fluid particle located at x was modelled by

$$\frac{d\mathbf{x}}{dt} = \mathbf{V}(\mathbf{x}, t) + \xi(t) \tag{1}$$

where $\mathbf{V}(\mathbf{x}, t)$ is the deterministic velocity field, and $\xi(t)$ is a Brownian motion which models the effects of molecular diffusion [19].

For chaotic flows, attention is restricted in this paper to Stokes flows, which are known to be incompressible flows. The following analysis holds for both two-dimensional time periodic flows, and three-dimensional stationary flows: in those two cases, *because of incompressibility*, there is a single positive Lyapunov exponent λ, and a single negative Lyapunov exponent equal to $-\lambda$. In the three-dimensional case, the third exponent is equal to zero. Indeed, when chaos is homogeneously global, λ does not depend on the initial location of the blob [20]. The effects of molecular diffusion are studied theoretically. The time at which the smallest scalar scale of the flow is reached is found, i.e. when the concentration gradient created by advection begin to be smoothed by diffusion. It is shown that diffusion is enhanced by chaotic advection. The energy dissipated in a Stokes flow is then calculated, and compared with the case when the flow is turbulent. In the last section, a new parameter is introduced that takes into account both the mixing time and the energy dissipated. The choice between laminar and turbulent mixing is discussed.

2. EFFECTS OF DIFFUSION IN STOKES FLOWS

2.1. *Dimensional analysis*

Using dimensional analysis, a search is first made for a general law for the time, τ_{diff}, at which molecular diffusion begins to act significantly. When the fluid is at rest, this time depends only on the thickness, L, of the blob to be mixed, and on the molecular diffusion coefficient, D. Using dimensional analysis gives

$$\tau_{\mathrm{diff}} \sim \frac{L^2}{D}. \tag{2}$$

In the presence of advection, this time will also depend on the velocity gradient, say $\alpha \sim V/L$, where V is a

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NOMENCLATURE

C	concentration field
D	molecular diffusion coefficient
e	base of natural logarithms
$E(k)$	energy spectrum
f	arbitrary function
$\mathbf{G}$	concentration gradient vector
i	subscript integer for a vector component, $i = 1, 2, 3$
I	parameter quantifying the mixing efficiency
j	subscript integer for a vector component, $j = 1, 2, 3$
k	wave number
l	size of a blob in the smallest direction
l_{island}	typical size of an unstirred island
L	characteristic length scale
Pe	Péclet number, VL/D
q^2	energy associated with the velocity fluctuations
Re	Reynolds number of the Stokes flow, VL/ν
Re_{turb}	turbulent Reynolds number, $\sqrt{q^2}L/\nu$
Sc	Schmidt number, ν/D
t	time
T	characteristic time-scale
T_L^η	time taken for a blob of size L to reach size η
$T_\eta^{\eta_c}$	time taken for a blob of size η to reach size η_c
V	characteristic velocity
$\mathbf{V}$	velocity vector
W_{Sto}	total energy dissipated per unit volume in the Stokes flow
W_{turb}	total energy dissipated per unit volume in the turbulent flow
$\mathbf{x}$	position vector.

Greek symbols

α	velocity gradient
ε	dissipation rate per unit mass
η	Kolmogorov scale, $L/Re_{\text{turb}}^{3/4}$
η_c	Batchelor scale, $\eta/\sqrt{Sc}$
λ	Lyapunov exponent
μ	viscosity of the fluid
ν	kinematic viscosity of the fluid, μ/ρ
ξ	Brownian motion vector
ρ	density of the fluid
$\tau(l)$	characteristic time based on an eddy of size l
Φ	power dissipated per unit volume in the Stokes flow
τ_{diff}	time when molecular diffusion begins to act significantly
$(\tau_{\text{diff}})_{\text{turb}}$	time when molecular diffusion becomes important in the turbulent flow
τ'_{diff}	mixing time with an unstirred island.

characteristic velocity of the Stokes flow. Using dimensional analysis, gives

$$\tau_{\text{diff}} \sim \frac{1}{\alpha} f\left(\frac{\alpha L^2}{D}\right) \sim \frac{L}{V} f\left(\frac{VL}{D}\right) \tag{3}$$

where f is a function that depends on the given flow-field. In this formulation, one can recognize the Péclet number, Pe, of the flow

$$Pe = \frac{VL}{D}. \tag{4}$$

This may also be rewritten using the Reynolds number, $Re = VL/\nu$, of the flow, and the Schmidt number, $Sc = \nu/D$, where ν is the kinematic viscosity of the fluid, as

$$\tau_{\text{diff}} \sim \frac{L^2}{\nu} \frac{1}{Re} f(Re\, Sc). \tag{5}$$

If the coefficient D decreases, Pe increases. Moreover, physically, the time when molecular diffusion becomes important increases as D decreases. Thus, f is an increasing function of Pe. Observe that the smaller f is, the less time it takes to mix. First suppose that the effects of advection are negligible. In this case, the result without advection given by (2) should be recovered: thus

$$f(Pe) = Pe \tag{6}$$

in equation (3). This result applies when the diffusive time, L^2/D, is much less than the convective time, L/V, that is, when

$$\frac{L^2/D}{L/V} = \frac{VL}{D} = Pe \ll 1. \tag{7}$$

Thus, equation (6) must hold when Pe is small. If advection is significant, i.e. Pe is not small, there is a function f which is a growing function of Pe, but which grows less rapidly than Pe since advection should enhance mixing, and hence decrease f. A classical result when the flow is laminar and non-chaotic is that τ_{diff} behaves like $Sc^{1/3}$ for large Pe, which implies that

$$f(Pe) = Pe^{1/3} \tag{8}$$

It is expected that f is further decreased when the flow is chaotic, and it will be shown later that this is the case.

2.2. *Effects of diffusion with chaos*

According to the Fick law, the effects of molecular diffusion become more important as the concentration gradient increases. Consider a blob of dye submitted to a steady strain: it becomes thinner and thinner in the direction of contraction, and longer and longer in the other direction. After a while, this blob no longer becomes thinner, since molecular diffusion enlarges it at the same rate as it is contracted by the strain: at that time, the concentration gradient, **G**, defined as

$$G_i = \frac{\partial C}{\partial x_i} \tag{9}$$

has a time derivative equal to zero. If we look at the equation of evolution of **G**:

$$\frac{\mathrm{d}G_i}{\mathrm{d}t} = -G_j\frac{\partial v_j}{\partial x_i} + D\Delta G_i \tag{10}$$

it is seen that the two terms on the right-hand side of equation (10) must then become of the same order of magnitude, and molecular diffusion becomes significant. As will be seen, the case of mixing in an incompressible chaotic flow is not different.

For analysis, it will be supposed that advection acts first, until the concentration gradients are great enough for diffusion to become non-negligible. This happens when the diffusive time is much greater than the convective time, i.e.

$$Pe = \frac{VL}{D} \gg 1. \tag{11}$$

Come back to equation (10). The first term on its right-hand side is the term for creation of concentration gradient by advection. It involves the *local* velocity gradient governing contraction, which, in a chaotic flow, is the most negative Lyapunov exponent [17], here $-\lambda$. The second term is the term for destruction of the concentration gradients by molecular diffusion. The two terms become of the same order of magnitude when $t = \tau_{\mathrm{diff}}$, which implies that

$$\lambda \sim \frac{D}{l^2(\tau_{\mathrm{diff}})} \tag{12}$$

where $l(t)$ is the smallest size of the blob after time t, i.e. its size in the direction of contraction, once again governed by the most negative Lyapunov exponent, $-\lambda$:

$$l(t) \sim L\exp(-\lambda t) \tag{13}$$

where it is supposed that the blob has an initial size L.

In order to satisfy equations (12) and (13)

$$\tau_{\mathrm{diff}} \sim \frac{1}{2\lambda}\ln\frac{\lambda L^2}{D} \tag{14}$$

is obtained. This result was already known to Ott and Antonsen [17], who had derived it from the evolution equation for the typical separation δ between two points. It shows that, for a given flow, the time τ_{diff} at which the effects of molecular diffusion compensate contraction by the flow-field varies as $\ln 1/D$, which is also in agreement with the scaling obtained by Aref and Jones [16].

In the derivation of equation (14), the hypothesis of a Stokes flow was not used. However, when comparing equation (3) with equation (14), it is obvious that λ is proportional to the mean velocity gradient in the flow. This may be explained as follows: in a Stokes flow, the equations of motion are linear. Therefore, changing the characteristic velocity V of the flow just changes the characteristic time-scale T, but not the topology of the trajectories: for example, if V is multiplied by 2, T is divided by the same factor, so as to keep the characteristic length scale VT constant. Now, by looking at equation (13), it can be seen that the product λT must also be kept constant in order to have the same length scales, which is also verified by multiplying λ by 2. The Lyapunov exponent is thus proportional to the mean velocity gradient. Note that this last result would not be available in an inviscid fluid because of the non-linearity of the equations for the flow-field. Thus, letting $\lambda \sim V/L$ in equation (14) gives

$$\tau_{\mathrm{diff}} \sim \frac{L}{2V}\ln\frac{VL}{D}. \tag{15}$$

This last equation implies that

$$f(Pe) = \frac{1}{2}\ln Pe. \tag{16}$$

When written using Re and Sc,

$$\tau_{\mathrm{diff}} \sim \frac{L^2}{2\nu}\frac{1}{Re}\ln(Re\,Sc) \tag{17}$$

is obtained. It is recalled that this result requires that $Pe \gg 1$, which gives

$$Re\,Sc \gg 1. \tag{18}$$

Figure 1 shows the variation of the mixing time τ_{diff} in a Stokes flow as a function of Re. It can be seen that the mixing time is a rapidly decaying function of Re as soon as $Re > \mathrm{e}/Sc$ (e being the base of natural logarithms), which includes the zone of validity, $Pe \gg 1$, of formula (17). When $Re = \mathrm{e}/Sc$ i.e. $Pe = \mathrm{e}$, which is the lower limit of equation (17), it follows that

$$\tau_{\mathrm{diff}} \sim \frac{L^2}{2\nu}\frac{Sc}{\mathrm{e}} \tag{19}$$

while at very small Re, $Pe \ll 1$, τ_{diff} is given by

$$\tau_{\mathrm{diff}} \sim \frac{L^2}{\nu}Sc \tag{20}$$

which agrees with equation (19) in order of magnitude.

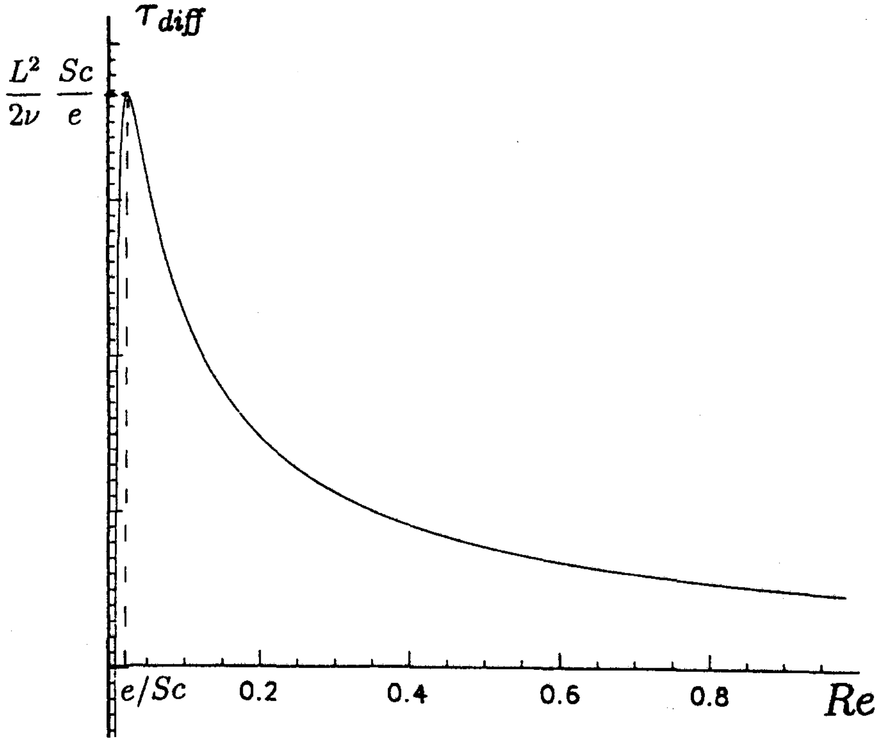

Fig. 1. Graph of τ_{diff} as a function of Re in a Stokes flow.

However, it is not at all common that a Stokes flow would mix perfectly on advective time-scales: a completely chaotic phase space is needed for this. Indeed, in many experiments, islands of poor stirring remain in the flow, which are quite difficult to get rid of (the choice of the parameters for efficient stirring is, in fact, the main difficulty in chaotic mixing). When chaos is global, the smallest length scale of the flow, $l(\tau_{\text{diff}})$, is given by equation (12) with $\lambda \sim V/L$:

$$l(\tau_{\text{diff}}) \sim \frac{L}{\sqrt{Pe}}. \tag{21}$$

This length scale might be very small at very high Pe. If the unstirred island has a typical size l_{island} which is bigger than that, it will get mixed on the diffusive time-scale τ'_{diff} given by equation (2) for $L = l_{\text{island}}$:

$$\tau'_{\text{diff}} \sim \frac{l^2_{\text{island}}}{D}. \tag{22}$$

Thus, τ'_{diff} might be much greater than τ_{diff}, so that good mixing could take much longer when unmixed islands remain in the chaotic flow. In the following, it will be supposed that the stirring is efficient, i.e. the phase space is completely chaotic, and τ_{diff} will be taken as the mixing time.

3. ENERGY SAVING

Compare two different situations of mixing of two fluids having the same density ρ and viscosity μ (passive tracer). In the first case, the velocities are very small (Stokes flow), but the trajectories are chaotic; in the second one, mixing is realized by making the flow turbulent. It is obvious that the ratio of the energy dissipated which comes into play is very large, but what about the ratio of required energies? This problem is studied in the case of a high Sc, which requires a very viscous fluid. Mixing is carried out in a domain having a characteristic length L, and, as before, the characteristic velocity in the Stokes flow is denoted by V. Thus, the Re of the Stokes flow is

$$Re = \frac{VL}{\nu} \ll 1. \tag{23}$$

As usual, the characteristic velocity in the turbulent flow is defined as the square-root of the mean energy associated with the velocity fluctuations, denoted by $\sqrt{q^2}$. Thus, the turbulent Re, Re_{turb}, is defined by

$$Re_{\text{turb}} = \frac{\sqrt{q^2}L}{\nu} \gg 1. \tag{24}$$

In both cases, mixing is achieved when molecular diffusion erases the concentration gradients created by advection, i.e. when the minimal *scalar* scale is reached.

3.1. *Energy dissipation in a chaotic Stokes flow*

The power Φ dissipated per unit volume in a Stokes flow is

$$\Phi \sim \mu \left(\frac{\partial U_i}{\partial x_j}\right)^2$$

$$\sim \mu \frac{V^2}{L^2} \tag{25}$$

The time τ_{diff} it takes for a blob of typical size L to stretch and shrink in width to the smallest size of the scalar field is given by equation (17).

The total energy, W_{Sto}, dissipated per volume unit is

$$W_{\mathrm{Sto}} \sim \Phi \tau_{\mathrm{diff}}$$

$$\sim \frac{\mu^2}{2\rho L^2} Re \ln(Re\, Sc). \tag{26}$$

3.2. *Energy dissipation in a turbulent flow*

3.2.1. *Power dissipated.* It is supposed that turbulence is such that the large structures have a typical size L. The power dissipated per unit volume is

$$\rho \bar{\varepsilon} \sim \rho \frac{(\overline{q^2})^{3/2}}{L}. \tag{27}$$

It is now worthwhile to estimate the time $(\tau_{\mathrm{diff}})_{\mathrm{turb}}$ when molecular diffusion becomes important, that is, the time when the smallest scalar scale is reached.

3.2.2. *Mixing time.* Firstly, a blob of size L is distorted during a time lapse T^{η}_{L} and reaches the smallest *velocity* length scale, the Kolmogorov scale, η, which, as usual, is

$$\eta \sim \frac{L}{Re_{\mathrm{turb}}^{3/4}}. \tag{28}$$

Then, it is further distorted to reach the smallest *scalar* length scale, the Batchelor scale, η_{c}, which, at high Sc, is given by

$$\eta_{\mathrm{c}} \sim \frac{\eta}{\sqrt{Sc}}. \tag{29}$$

In order to calculate the first contribution T^{η}_{L}, the reasoning done by Broadwell and Briedenthal [21] is used, which supposes that a blob of size l ($\eta \leqslant l \leqslant$ L) is mainly distorted by the eddies of the turbulent flow that have the same size l: eddies of size greater than l merely convect it, and those that have a size smaller than l do not affect it. Thus:

$$-\frac{1}{l}\frac{\mathrm{d}l}{\mathrm{d}t} \sim \frac{1}{\tau(l)} \tag{30}$$

where $\tau(l)$ is the characteristic time based on an eddy of size l. Such an eddy has an energy per unit mass of order $q^2(l)$, so that

$$\frac{1}{\tau(l)} \sim \frac{\sqrt{q^2(l)}}{l} \sim \frac{1}{l}\sqrt{kE(k)}$$

$$\sim l^{-3/2} E^{1/2}(k) \tag{31}$$

where $k \sim 1/l$ is the wave number, and $E(k)$ is the energy spectrum. Since it is supposed that $Re_{\mathrm{turb}} \gg 1$, in the inertial range of scales between η and L the Kolmogorov spectrum

$$E(k) \sim \bar{\varepsilon}^{2/3} k^{-5/3} \sim \frac{\overline{q^2}}{L^{2/3}} l^{5/3} \tag{32}$$

which, when combined with equation (31), gives

$$\frac{1}{\tau(l)} \sim \frac{\sqrt{\overline{q^2}}}{L^{1/3}}(l)^{-2/3}. \tag{33}$$

Integration of equation (3) between $l = L$ and $l = \eta$ then gives

$$T^{\eta}_{\mathrm{L}} \sim \frac{L}{\sqrt{\overline{q^2}}}\left[1 - \left(\frac{\eta}{L}\right)^{2/3}\right]$$

$$\sim \frac{L}{\sqrt{\overline{q^2}}}\left(1 - \frac{1}{\sqrt{Re_{\mathrm{turb}}}}\right). \tag{34}$$

This reasoning is extended in the range $\eta_{\mathrm{c}} \leqslant l \leqslant \eta$, by supposing that the eddies that affect the scalar field are the smallest eddies, i.e. those having the size of the Kolmogorov microscale, η. Thus, letting $l = \eta$ in equation (30), and using equations (33) and (28), gives

$$-\frac{1}{l}\frac{\mathrm{d}l}{\mathrm{d}t} \sim \frac{\sqrt{\overline{q^2}}}{L^{1/3}}\eta^{-2/3} \sim \frac{\sqrt{\overline{q^2}}}{L}\sqrt{Re_{\mathrm{turb}}}. \tag{35}$$

Integration between $l = \eta$ and $l = \eta_{\mathrm{c}}$ gives

$$T^{\eta}_{\eta_{\mathrm{c}}} \sim \frac{L}{\sqrt{\overline{q^2}}}\frac{1}{\sqrt{Re_{\mathrm{turb}}}}\ln\frac{\eta}{\eta_{\mathrm{c}}}. \tag{36}$$

This may be rewritten using equation (29) as

$$T^{\eta}_{\eta_{\mathrm{c}}} \sim \frac{L}{\sqrt{\overline{q^2}}}\frac{1}{\sqrt{Re_{\mathrm{turb}}}}\ln\sqrt{Sc}. \tag{37}$$

The total time $(\tau_{\mathrm{diff}})_{\mathrm{turb}} = T^{\eta}_{\mathrm{L}} + T^{\eta}_{\eta_{\mathrm{c}}}$, for a blob of size L to reach the Batchelor microscale, η_{c} is therefore

$$(\tau_{\mathrm{diff}})_{\mathrm{turb}} \sim \frac{L}{\sqrt{\overline{q^2}}}\left[1 + \frac{1}{\sqrt{Re_{\mathrm{turb}}}}(\ln\sqrt{Sc} - 1)\right]. \tag{38}$$

Note that, at very high Re, $(\tau_{\mathrm{diff}})_{\mathrm{turb}}$ no longer depends on the Sc, which is in agreement with the analysis given by Broadwell and Breidenthal [21].

3.2.3. *Energy dissipated.* From equations (38) and (27), the energy dissipated per unit volume of domain for the mixing process in a turbulent flow is obtained:

$$W_{\mathrm{turb}} \sim \rho\bar{\varepsilon}(\tau_{\mathrm{diff}})_{\mathrm{turb}}$$

$$\sim \frac{\mu^2}{\rho L^2} Re_{\mathrm{turb}}^2 \left(1 + \frac{\ln\sqrt{Sc} - 1}{\sqrt{Re_{\mathrm{turb}}}}\right) \tag{39}$$

3.3. *Energy saving*

Compare the ratio of the energy dissipated in the Stokes flow and the turbulent flow, given, respectively, by equations (26) and (39):

$$\frac{W_{\mathrm{Sto}}}{W_{\mathrm{turb}}} \sim \frac{Re \ln(Re\, Sc)}{2Re_{\mathrm{turb}}^2\left(1+\dfrac{\ln\sqrt{Sc}-1}{\sqrt{Re_{\mathrm{turb}}}}\right)} \tag{40}$$

Now, it is supposed that $Re \ll 1$, but that $Pe = ReSc \gg 1$. Thus, the logarithms in formula (40) have large arguments. However, the logarithm is a very weakly growing function at large arguments, and the dominant behaviour is

$$\frac{W_{\mathrm{Sto}}}{W_{\mathrm{turb}}} \approx \frac{Re}{2Re_{\mathrm{turb}}^2} \tag{41}$$

which is small because of equations (23) and (24). Thus, a chaotic Stokes flow requires much less energy than a turbulent flow.

4. THE ENGINEERING POINT OF VIEW

To sum up the situation: in the first case, the flow is laminar, and the energy dissipated to achieve the mixing is low. In the second case, the flow is turbulent, and the energy dissipated is large. However, the ratio of the mixing times for the two cases, given, respectively, by equations (17) and (38), is

$$\frac{\tau_{\mathrm{diff}}}{(\tau_{\mathrm{diff}})_{\mathrm{turb}}} \sim \frac{Re_{\mathrm{turb}}}{2Re} \frac{\ln(Re\, Sc)}{1+\dfrac{\ln\sqrt{Sc}-1}{\sqrt{Re_{\mathrm{turb}}}}}$$

$$\approx \frac{1}{2}\frac{Re_{\mathrm{turb}}}{Re} \gg 1 \tag{42}$$

The Stokes flow takes much longer to effect mixing. Which is best? Is the fact that the energy dissipated is much less in the Stokes flow sufficient if the mixing time is very long?

4.1. *A new parameter for mixing*

It is proposed to introduce another parameter to quantify the efficiency of mixing, denoted by I. It is defined to be the product of the energy dissipated and the mixing time

$$I \sim W\tau_{\mathrm{diff}}. \tag{43}$$

Mixing is most efficient when I is at its minimum.

In the Stokes flow, from equations (17) and (26):

$$I_{\mathrm{Sto}} \sim \frac{\mu}{4}\ln^2(Re\, Sc). \tag{44}$$

Owing to the slow growth of the logarithm function, I_{Sto} depends only weakly on the Reynolds number of the flow, and is nearly a characteristic of the fluid. In the case of turbulent mixing, from equations (38) and (39):

$$I_{\mathrm{turb}} \sim \mu\, Re_{\mathrm{turb}}\left(1+\frac{\ln\sqrt{Sc}-1}{\sqrt{Re_{\mathrm{turb}}}}\right)^2 \tag{45}$$

and I_{turb} is roughly proportional to the Re of the flow. This reflects the fact that increasing Re too much is wasteful, since the energy lost is much greater than the reduction in mixing time.

4.2. *Comparison of the Stokes flow and the turbulent flow*

The ratio of I in the Stokes flow to that in the turbulent flow is

$$\frac{I_{\mathrm{Sto}}}{I_{\mathrm{turb}}} \sim \frac{1}{4Re_{\mathrm{turb}}}\left[\frac{\ln(Re\, Sc)}{1+\dfrac{\ln\sqrt{Sc}-1}{\sqrt{Re_{\mathrm{turb}}}}}\right]^2. \tag{46}$$

Now consider the mixing of a contaminant in a very viscous fluid, for which molecular diffusion is weak: Sc is, say, 10^{12} (fluorescent contaminant in glycerine) [22]. Suppose that the mixing can be achieved for a slightly supercritical turbulent flow, for which Re_{turb} might be, say 100 (this is actually the lower limit for a turbulent flow): The aim is to compare the ratio $I_{\mathrm{Sto}}/I_{\mathrm{turb}}$ when the Stokes flow is such that its Re is equal to unity (this is indeed the upper limit for a Stokes flow). In those extreme conditions:

$$\frac{I_{\mathrm{Sto}}}{I_{\mathrm{turb}}} \sim 0.37 < 1. \tag{47}$$

Thus, it can be seen that laminar mixing is more efficient than turbulent mixing, even judged by the parameter I, which includes the energy and time taken on an equal footing. However, it must be observed once again that this result only applies when Sc is very large, since the mixing time in the Stokes flow was calculated using hypothesis (11) which, together with equation (23), implies that

$$\frac{1}{Sc} \ll Re \ll 1. \tag{48}$$

Sc must be at least of order 10^4 (say), which likely happens in very viscous fluids. Moreover, it has previously been supposed that chaos was global in the flow, so that stirring was efficient. When this is not the case, the mixing time [given by equation (22)] might be much longer, and, because of this, the recalculated ratio $I_{\mathrm{Sto}}/I_{\mathrm{turb}}$ could be larger than unity. The importance of efficient stirring is indeed crucial in chaotic mixing.

5. SUMMARY AND CONCLUSION

In this paper, mixing in chaotic Stokes flows and turbulent flows has been compared. The mixing time in a chaotic laminar flow has been calculated theoretically, defined as the time when the concentration gradients created by advection are smoothed by the molecular diffusion. It is found that this time varies with Sc as ln (Sc), which shows that chaotic advection greatly enhances the effects of diffusion. However, despite the smoothing effects of the logarithm function, this time may be quite long at high Sc. The powers dissipated in a turbulent flow and a Stokes flow were then calculated, which, when multiplied by the mixing time give the energy dissipated in both cases. On the one hand, mixing in turbulent flows is very rapid compared to chaotic flows, but, on the other hand, chaotic mixing is much more energy-saving. The question which arises is thus the following: is it better to mix by making the flow turbulent (energy-wasting but very rapid) or by a Stokes flow with chaotic trajectories (energy-saving but quite long)? It is proposed that one compares the product of the energy dissipated by the mixing time in the two cases, and chooses the lowest: it is shown that, at a very high Sc ($\sim 10^4$ at least), mixing in globally chaotic Stokes flow is more efficient than turbulent mixing, and this result is to be taken into account in the case of very viscous fluids.

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